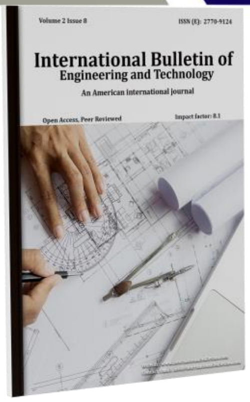

HIGHER ALGEBRA IN COMBINATORICS

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Abstract: This article has easy and a very nice application on how to solve elementary combinatorics problems using linear algebra

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We have seen numerous applications of analysis and higher algebra in number theory and algebra. It is time to see the contribution of this ”non-elementary mathematics” to combinatorics. It is quite hard to imagine that behind a simple game, such as football for example or behind a quotidian situation such as handshakes there exists such complicated machinery, but this happens sometimes and we will prove it in the next. For the beginning of the discussion, the reader doesn't need any special knowledge, just imagination and the most basic properties of the matrices, but, as soon as we advance, things change. Anyway, the most important fact is not the knowledge, but the ideas and, as we will see, it is not easy to discover that ”non-elementary” fact that hides after a completely elementary problem. Since we have clarified what is the purpose of the unit, we can begin the battle.

The first problem we are going to discuss is not classical, but it is easy and a very nice application of how linear-algebra can solve elementary problems. Here it is.

Example 1. Let $n \geq 3$ and let A_n, B_n be the sets of all even, respectively, odd permutations of the set $\{1,2,\ldots, n\}$. Prove the equality

$$\sum_{\sigma\in A_n}\sum_{i=1}^{n}|i-\sigma(i)| = \sum_{\sigma\in B_n}\sum_{i=1}^{n}|i-\sigma(i)|$$

Solution. Writing the diffference

$$\sum_{\sigma\in A_n}\sum_{i=1}^{n}|i-\sigma(i)| - \sum_{\sigma\in B_n}\sum_{i=1}^{n}|i-\sigma(i)|$$

as

$$\sum_{\sigma\in S_n}\varepsilon(\sigma)\sum_{i=1}^{n}|i-\sigma(i)| = 0,$$

Where

$$\varepsilon(\sigma) = \begin{cases} 1, \text{if } \sigma \in A_n \\ -1, \text{if } \sigma \in B_n \end{cases}$$

reminds us about the formula

$$\det A = \sum_{\sigma\in S_n}\varepsilon(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}$$

We have taken here $S_n = A_n \cup B_n$. But we don't have any product in our sum! That is why we will take an arbitrary positive number a and we will consider the matrix $A = (a|i-j|)1 \le i,j \le n$. This time,

$$\det A = (-1)^{\varepsilon(\alpha)} a^{|1-\sigma(1)|} \ldots a^{|-n\sigma(n)|} = \sum_{\sigma \in A_n} a^{\sum_{i=1}^{n}|i-\sigma(i)|} - \sum_{\sigma \in B_n} a^{\sum_{i=1}^{n}|i-\sigma(i)|}$$

This is how we have obtained the identity

$$\begin{vmatrix} 1 & \cdots & x^{n-1} \\ \vdots & \ddots & \vdots \\ x^{n-1} & \cdots & 1 \end{vmatrix}$$

Anyway, we still do not have the desired difference. What can we do to obtain it? The most natural way is to derive the last relation, which is nothing else than a polynomial identity, and then to take $x = 1$. Before doing that, let us observe that the polynomial

$$\begin{vmatrix} 1 & \cdots & x^{n-1} \\ \vdots & \ddots & \vdots \\ x^{n-1} & \cdots & 1 \end{vmatrix}$$

is divisible by $(x - 1)^2$. This can be easily seen by subtracting the first line from the second and the third one and taking from each of these line $x - 1$ as common factor. Thus, the derivative of this polynomial is a polynomial divisible by $x - 1$, which shows that after we derive the relation (1) and take x = 1, in the left-hand side we will obtain 0. Since in the right-hand side we obtain exactly

$$\sum_{\sigma \in A_n} \sum_{i=1}^{n} |i - \sigma(i)| - \sum_{\sigma \in B_n} \sum_{i=1}^{n} |i - \sigma(i)|$$

the identity is established.

Here is another nice application of this trick. We have seen how many permutation do not have a fixed point. The question that arises is how many of them are even. Here is a direct answer to the question, using determinants.

Example 2. Find the number of even permutations of the set $\{1, 2, \ldots, n\}$ that do not have fixed points.

Solution. Let us consider C_n, D_n, respectively, the sets of even and odd permutations of the set $\{1, 2, \ldots, n\}$, that do not have any fixed points. We know how to find the sum $|C_n| + |D_n|$. We have seen it is equal to

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right).$$

Hence if we manage to compute the difference $|C_n| + |D_n|$, will be able to answer to the question. If we write

$$|C_n| - |D_n| = \sum_{\substack{\sigma \in A_n \\ \sigma(i) \neq i}} 1 - \sum_{\substack{\sigma \in B_n \\ \sigma(i) \neq i}} 1,$$

we observe that this reduces to computing the determinant of the matrix $T = (t_{ij})_{1 \le i,j \le n}$, where

$$t_{ij} = \begin{cases} 1, \text{if } i \neq j \\ 0, \text{if } i = j \end{cases}$$

That is,

$$|C_n| - |D_n| = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

But it is not difficult to compute this determinant. Indeed, we add all columns to the first one and we give $n - 1$ as common factor, then we subtract the first column from each of the other columns. The result is $|C_n| - |D_n| = (-1)^{n-1}(n-1)$ and the conclusion is quite surprising:

$$|C_n| = \frac{1}{2}n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n-2}}{(n-2)!}\right) + (-1)^{n-1}(n-1).$$

We will focus in the next problems on a very important combinatorial tool, that is the incidence matrix (cum se spune la matricea de incidenta?). What is this? Suppose we have a set $X = \{x_1, x_2, \dots, x_n\}$ and $X_1, X_2, \dots, X_k$ a family of subsets of X. Now, define the matrix $A = (a_{ij})_{\substack{i=1,n,\\ j=1,k}}$

where

$$a_{ij} = \begin{cases} 1, \text{if } x_i \in X_j \\ 0, \text{if } x_j \notin X_j \end{cases}$$

This is the incidence matrix of the family $X_1, X_2, \dots, X_k$ and the set X. In many situations, computing the product ${}^tA \cdot A$ helps us to modelate algebraically the conditions and the conclusions of a certain problem. From this point, the machinery activates and the problem is on its way of solving.

Let us discuss first a classical problem, though a diffiffifficult one. It appeared in USAMO 1979, Tournament of the Towns 1985 and in Bulgarian Spring Mathematical Competition 1995. This says something about the classical character and beauty of this problem.

Example 3. Let $A_1, A_2, \dots, A_{(n+1)}$ be distinct subsets of the set $\{1, 2, \dots, n\}$, each of which having exactly three elements. Prove that there are two distinct subsets among them that have exactly one point in common.

Solution. Of course, we argue by contradiction and suppose that $|A_i \cap A_j| \in \{0,2\}$ for all $i \neq j$. Now, let T be the incidency matrix of the family $A_1, A_2, \dots, A_{n+1}$ and compute the product

$${}^tT \cdot T = \begin{pmatrix} \sum_{k=1}^{n} t_{k_1}^2 & \cdots & \sum_{k=1}^{n} t_k t_{kn+1} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} t_{kn+1} t_{k_1} & \cdots & \sum_{k=1}^{n} t_{kn+1}^2 \end{pmatrix}$$

But we have of course

$$\sum_{k=}^{n} x_{ki}^2 = |A_i| = 3$$

and

$$\sum_{k=1}^{n} x_{ki} x_{kj} = |A_i \cap A_j| \in \{0,2\}$$

Thus, considered in the fifield $(R_2, +, \cdot)$, we have

$$\overline{{}^{t}T \cdot T} = \begin{pmatrix} \hat{1} & \cdots & \hat{0} \\ \vdots & \ddots & \vdots \\ \hat{0} & \cdots & \hat{1} \end{pmatrix}$$

where $\overline{X}$is the matrix having as elements the residues classes of the elements of the matrix X. Since of course $\det \overline{X} = \overline{\det X}$, the previous relation shows that $\det {}^{t}T \cdot T$is odd, hence non-zero. This means that ${}^{t}T \cdot T$ is an invertible matrix of size $n + 1$, thus $\operatorname{rank}({}^{t}T \cdot T) = n + 1$ which contradicts the inequality $\operatorname{rank}({}^{t}T \cdot T) \leq \operatorname{rank}T \leq n$. This shows that our assumption was wrong and there exist indeed indices $i \neq j$ such that $|A_i \cap A_j| = 1$

The following problem is very difficult to solve by elementary means, but the solution using linear-algebra is straightforward.

Example 4. Let n be an even number and $A_1, A_2, \ldots, A_{(n+1)}$ be distinct subsets of the set $\{1, 2, \ldots, n\}$, each of them having an even number of elements. Prove that among these subsets there are two having an even number of elements in common.

Solution. Indeed, if T is the incidency matrix of the family $A_1, A_2, \ldots, A_{(n+1)}$, we obtain as in the previous problem the following rela tion

$${}^{t}T \cdot T = \begin{pmatrix} |A_1| & \cdots & |A_1 \cap A_n| \\ \vdots & \ddots & \vdots \\ |A_n \cap A_1| & \cdots & |A_n| \end{pmatrix}$$

Now, let us suppose that all the numbers $|A_i \cap A_j|$are odd and interpret the above relation in the field $(R_2, +, \cdot)$. We find that which means again that $\det {}^{t}T \cdot T$is odd. Indeed, if we work in $(R_2, +, \cdot)$, we obtain

$$\begin{vmatrix} \hat{0} & \cdots & \hat{1} \\ \vdots & \ddots & \vdots \\ \hat{1} & \cdots & \hat{0} \end{vmatrix} = \hat{1}$$

The technique used is exactly the same as in the second example,only this time we work in a different field. Note that this is the moment when we use the hypothesis that n is even. Now, since $\det {}^{t}T \cdot T = \det^2 T$, we obtain that $\det T$is also an odd number. Hence we should try to prove that in fact $\det T$is an even number and the problem will be solved. Just observe that the sum of elements of the column i of T is $|A_i|$,hence an even number. Thus, if we add all lines to the first line, we will obtain only even numbers on the first line. Since the value of the determinant doesn't change under this operation, the conclusion is plain: $\det T$is an even number. Since a number cannot be both even and odd, our assumption was wrong and the problem is solved.

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